# THE DIRECT METHOD OF LIAPUNOV APPLIED TO THE DESIGN OF CONTROLLERS FOR A CLASS OF NONLINEAR AND TIME VARYING PROCESSES

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# THE DIRECT METHOD OF LIAPUNOV APPLIED TO THE DESIGN OF CONTROLLERS FOR A CLASS OF NONLINEAR AND TIME VARYING PROCESSES*

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Summary

Liapunov's direct method is employed in the design of intentionally nonlinear controllers for a class of nonlinear, time varying processes. The criterion of asymptotic stability is applied to the equation for the error between the process output and a desired output obtained from a model reference. A control signal generated on the basis of the error continually forces the process output to approach the desired output. The design procedure is developed for processes containing a single time varying, nonlinear gain element as well as linear time varying parameters. Restrictions on the nonlinear function are derived. In a general example the design procedure is applied to a second order process containing a randomly switching nonlinear gain element and also a time varying right half plane pole. A second example, in which practical design considerations are discussed, involves the design of a controller for a conditionally stable process. Analogue computer results presented for both examples are in good agreement with theoretical results.

Author

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## I. INTRODUCTION

The direct method of Liapunov has been employed in the design of intentionally nonlinear controllers for linear time varying processes[1]. The design is based on comparing the processes output to a model reference which has the desired output. A control signal generated on the basis of the comparison forces the difference between model and process output to be zero. The design procedure insures asymptotic stability of the resultant system.

In this paper the design procedure discussed above is extended to processes of the type shown in Fig. 1., i.e. processes containing a single time varying, nonlinear gain element as well as linear time varying parameters. Restrictions on $f(y,t)$ are as indicated in (2) and (27). The general form of $f(y,t)$ is as indicated in Fig. 2.

The nonlinear controller in Fig. 1 generates a controlled input, $u(t)$, which can be a function of any or all of the signals $c$, $c_d$, $e$, $r$, or their derivatives. This controlled input is such that it forces the time derivative of a positive definite Liapunov function, $V(e,\dot{e})$ to be negative definite. Consequently, the equilibrium state corresponding to $e = \dot{e} = 0$ is asymptotically stable in the large, i.e. the process output must eventually equal the model out under any initial conditions.

A general example to illustrate the design procedure is given in which $f(y,t)$ randomly assumes the two forms of Fig. 3a, while $a(t)$ of Fig. 1 switches randomly between 0 and $-1$.

Another example of more practical interest is given in which a controller is designed for the conditionally stable process illustrated in Fig. 9. Without the controlled input, $u(t)$, the process is unstable for large step inputs, $r(t)$. The presence of the controller makes the system stable for any magnitude of step input.

Analogue computer results for both examples are presented which demonstrate the effectiveness of the design. Step responses and trajectories in the $e, \dot{e}$ plane are shown for the general example. Phase plane trajectories are given for the example involving the conditionally stable system.

## II. THEORY

The process to be controlled in Fig. 1 is described by the equation

$$\ddot{c} + a(t)\,\dot{c} = f(y,t) \tag{1}$$

where $f(y,t)$, as shown in Fig. 2, is a nonlinear gain element which satisfies the conditions:

$$f\,(0,t) = 0 \tag{2a}$$

$$\left.\frac{\partial f}{\partial y}\right|_{y\,=\,0} = K(t) \tag{2b}$$

Let $K(t)$ and $a(t)$ be expressed as positive constants plus time varying parts as follows:

$$K(t) = K_o + k(t)\ ;\quad K_o > 0 \tag{3}$$

and

$$a(t) = a_o + \alpha(t)\ ;\ a_o > 0 \tag{4}$$

It is also convenient to express the nonlinear function as:

$$f(y,t) = \left\{K(t) - \phi\,(y,t)\right\}\,y \tag{5}$$

As a consequence of (2b) , $\phi\,(y,t)$ must satisfy the conditions:

$$\phi(0,t) = 0 \tag{6a}$$

$$\lim_{y\to 0}\ y\,\frac{\partial\,\phi(y,t)}{\partial\,y} = 0 \tag{6b}$$

If (3), (4) and (5) are substituted into (1), the resulting equation is

$$\ddot{c} + a_o\dot{c} = K_o y + \left\{k(t) - \phi(y,t)\right\}\,y - \alpha\,(t)\,\dot{c} \tag{7}$$

In order to obtain a constant coefficient term in the output variable c on the left hand side of (7), unity linear feedback around the process is introduced so that

$$y = r + u - c \tag{8}$$

where r is the process input, and u the controlled input.

Thus, (7) now becomes

$$\ddot{c} + a_o\dot{c} + K_c\ c = K_o\ (r+u) + \left\{k(t) - \phi(y,t)\right\} (r+u-c) - \alpha\,(t)\dot{c} \tag{9}$$

The model behavior is described by the equation

$$\ddot{c}_d + a_o\ \dot{c}_d + K_o c_d = K_o r \tag{10}$$

The error between the desired output, $c_d$ , and the process output, c , is defined as

$$e = c_d - c \tag{11}$$

The equation in the error variable, obtained by subtracting (9) from (10) is

$$\ddot{e} + a_o\dot{e} + K_o e = \left\{\phi(y,t) - K(t)\right\} u + \left\{\phi(y,t) - k(t)\right\} (r-c) + \alpha\,(t)\dot{c} \tag{12}$$

For the special conditions

$$\alpha\,(t) = 0 \tag{13a}$$

$$f(y,t) = K_o y \tag{13b}$$

(12) reduces to

$$\ddot{e} + a_o\ \dot{e} + K_o\ e\ =\ -K_o\ u \tag{14}$$

Conditions (13a) and (13b) imply that model and process are identical. Under these conditions, $\underline{c}_d$ and $\underline{c}$ will be identical only if the control signal is zero. Therefore, the requirement is placed on u that

$$u = o \text{ for } \underline{c}_d = \underline{c} \tag{15}$$

where the underline signifies vector notation, i.e. $\underline{c} = \begin{bmatrix} c \\ \dot{c} \end{bmatrix}$

With this requirement on u (14) reduces to the autonomous linear equation

$$\underline{\dot{e}} = A\underline{e} \tag{16}$$

where

$$e = \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{17}$$

and

$$A = \begin{bmatrix} o & 1 \\ -K_o - & a_o \end{bmatrix} \tag{17b}$$

For convenience, (12) is put into the form

$$\underline{\dot{e}} = A\underline{e} + \underline{h} \tag{18}$$

where

$$\underline{h} = \begin{bmatrix} 0 \\ \left\{\phi(y,t) - K(t)\right\} u + \left\{\phi(y,t) - k(t)\right\}(r-c) + \alpha(t)\dot{c} \end{bmatrix} \tag{19}$$

The point $\underline{e}$ = o represents an equilibrium state of (16). Since (16) is identical in form to the autonomous model equation

$$\underline{c_d} = A\underline{c_d} \tag{20}$$

then $\underline{e}$ = o is an asymptotically stable equilibrium point if a stable model is used. In the further development of the theory, it is assumed that this is the case.

Since (16) is a linear autonomous equation, theorem I (see Appendix) is applicable. The theorem is conveniently applied to find a Liapunov function for (16). Since $\underline{e}$ = o is asymptotically stable, then the theorem assures that there are symmetric positive definite matrices P and Q such that

$$A^T P + PA = -Q \tag{21}$$

and also that

$$V(\underline{e}) = \underline{e}^T P \, \underline{e} \tag{22}$$

is a Liapunov function for (16). This same function is used to investigate the stability properties of (18) with the idea in mind that it is to be made a Liapunov function of that equation by suitable choice of u.

The time derivative of (22) is

$$V(\underline{e}) = \dot{\underline{e}}^T P \, \underline{e} + \underline{e} \, P \, \dot{\underline{e}} \tag{23}$$

Use of (18) in (23) leads to

$$\dot{V}(\underline{e}) = \underline{e}^T (A^T P + PA) \, \underline{e} + 2\underline{e}^T P \underline{h} \tag{24}$$

The first term on the right hand side of (24) is negative definite because of (21).* If the second term can be made less than or equal to zero by choice of the controlled input u, then $\dot{V}(\underline{e})$ will be negative definite. Consequently, $V(\underline{e})$ (and also $\underline{e}$) must then approach zero, and u will accomplish the desired result of forcing the process output to follow the model output.

In order to determine the required u, the second term of (24) is expanded as follows:

$$2\underline{e}^T P \underline{h} = -2(p_{12}e_1 + p_{22}e_2) \left\{ \frac{f(y,t)}{y} \right\} \left\{ u - \frac{\{\phi(y,t) - k(t)\} \ (r-c) + \alpha(t)\dot{c}}{\frac{f(y,t)}{y}} \right\} \tag{25}$$

* This indicates the convenience of choosing (22) as a potential Liapunov function for (18).

where $p_{12}$ and $p_{22}$ are elements of the P matrix and $f(y,t)/y$ has been used in place of $\left\{K(t) - \phi(y,t)\right\}$. To insure that this term remains zero or negative, u is chosen as:

$$u = \left\{ \left| \frac{\{\phi(y,t) - k(t)\}(r-c)}{f(y,t)/y} \right|_{max} + \left| \frac{\alpha(t)\dot{c}}{f(y,t)/y} \right|_{max} \right\} \text{sat } b\gamma \tag{26a}$$

where $$\gamma = p_{12}\ e_1 + p_{22}\ e_2 \tag{26b}$$

and $$\text{sat } b\gamma = \begin{cases} +1 & \text{for } \gamma > 1/b \\ b & \text{for} -1/b < \gamma < 1/b \\ -1 & \text{for } \gamma < -1/b \end{cases} \tag{26c}$$

The saturation function (sat $b\gamma$ ) is used rather than sign $\gamma$ to satisfy (15) and to avoid problems involving the existence of solutions of (12)[2]. The saturation function is zero for $\gamma = 0$, but the sign function is undefined for $\gamma = 0$.

The controlled input u(t) as given in (26) is bounded and assures a negative definite $\dot{V}(\underline{e})$ if

$$\infty > \frac{f(y,t)}{y} > 0 \tag{27}$$

i.e. f(y,t) lies in the first and third quadrants. That f(y,t)/y must be positive is seen from (25). Since it appears in the denominator terms of (25) and (26), it cannot be zero if u is to be a bounded control signal. It is also clear from (26) that $\phi(y,t)$ , k(t) and $\alpha(t)$ must be bounded if u is to be finite. In view of (5), the conditions $|\phi(y,t)| < \infty$ , and $|k(t)| < \infty$ , lead to the condition $f(y,t)/y < \infty$.

These theoretical results will now be applied to a general example and to a specific design example.

## III. GENERAL EXAMPLE:

The following example pertains to the system of Fig. 1 with $K_o = a_o = 2$. The range of variation of the nonlinear functions, $f(y,t)$ and $\phi(y,t)$, is shown in Fig. 3. The pole at $s = -a(t)$ varies from $s = o$ into the right half plane to $s = +1$. In accordance with the model used, $K(t)$ and $a(t)$ are chosen as:

$$K(t) = 2 + k(t) \tag{28a}$$

$$a(t) = 2 + \alpha(t) \tag{28b}$$

where

$$-1.6 \leq k(t) \leq 0 \tag{28c}$$

and

$$-3 \leq \alpha(t) \leq -2 \tag{28d}$$

Fig. 3b illustrates that $\left|\phi(y,t) - k(t)\right| \leq 1.96.$

Use of (9) leads to the process equation

$$\ddot{c} + 2\dot{c} + 2c = 2(r+u) + \left\{k(t) - \phi(y,t)\right\}\left\{r + u - c\right\} - \alpha(t)\,\dot{c} \tag{29}$$

Since the model behavior is described by

$$\ddot{c}_d + 2\dot{c}_d + 2c_d = 2r \tag{30}$$

the error equation found by subtracting (29) from (30) is

$$\ddot{e} + 2\dot{e} + 2e = \left\{\phi(y,t) - K(t)\right\}u + \left\{\phi(y,t) - k(t)\right\}(r-c) + \alpha(t)\,\dot{c} \tag{31}$$

This equation may be written in matrix form as:

$$\underline{\dot{e}} = A\underline{e} + \underline{h} \tag{32}$$

The procedure discussed in conjunction with (16) and (18) is applied to (32). For convenience, Q of (21) is chosen to be the identity matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{33}$$

Then the solution of (21) for P is

$$P = 1/8 \begin{bmatrix} 10 & 2 \\ 2 & 3 \end{bmatrix} \tag{34}$$

Thus, for (32) the Liapunov function is

$$V(\underline{e}) = 5/4 \quad e_1^{\;2} + 1/2 \; e_1 \; e_2 + 3/8 \quad e_2^{\;2} \tag{35}$$

and

$$\dot{V}(\underline{e}) = -e^T e - \frac{1}{4}(2e_1 + 3e_2)\left\{\frac{f(y,t)}{y}\right\}\left\{u - \frac{\{\phi(y,t)-k(t)\}(r-c) + \alpha(t)\dot{c}}{f(y,t)/y}\right\} \tag{36}$$

From (26) the required control signal is

$$u = \left\{\left|48.9(r-c)\right| + \left|75\dot{c}\right|\right\} \text{sat } b \; (2e_1 + 3 \; e_2) \tag{37}$$

where b was chosen equal to 100.

The resultant system is illustrated in Fig. 4. Fig. 5 shows analogue computer results for c, $c_d$ and e. These results were taken while $\alpha(t)$ and k(t) were varying in a random step fashion between the limits indicated in (28c) and (28d) and shown in Fig. 3. It is seen that c and $c_d$ are in close agreement.

The computer results in Fig. 6 demonstrate the action of the control signal in terms of $V(\underline{e})$. Projections of the curves $V(\underline{e})$ = constant on the $e_1$, $e_2$ plane are shown along with trajectories of $e_1(t)$ vs. $e_2(t)$. For the second order system under consideration, all initial conditions are zero. Therefore, with the control signal present $V(\underline{e}) \equiv 0$. In order to obtain the results shown, the system was allowed to run uncontrolled until $t_1$, $t_2$, etc. at which times the control signal was applied ($t = 0$ signifies the origin and is the time system was started by applying a two volt step input). It can be seen that all trajectories are such that $\dot{V}(\underline{e}) < 0$ after the control signal is applied. For $t \to \infty$, all trajectories approach the origin along the switching line $e_2 = -2/3\, e_1$.

Another observation is that trajectories for Fig. 6a initially take a more leisurely approach to the switching line than do those of Fig. 6b. This is due to the fact that gain factors in the equation for u, (37), are exactly those required for conditions in case a and larger than required for conditions in case b. Also, the multiplying factor $f(y,t)/y$ in the equation for $\dot{V}(e)$, (36), is larger in case b than in case a, and this results in a more negative $\dot{V}(\underline{e})$.

## IV. DESIGN EXAMPLE FOR A CONDITIONALLY STABLE PROCESS

The control system shown in Fig. 7 is conditionally stable. For inputs greater than approximately 2 volts, it is seen from the phase plane plot of Fig. 8 that the trajectories go to infinity. Use of the control technique under discussion is illustrated in Fig. 9. The model need not be realized as shown, but is depicted as is so that signals within the model and process defined by x and z may be compared.

The model behavior is given by (30) and the system equation is

$$\ddot{c} - \dot{c} = f(y) = \left\{2-\phi(y)\right\} y \tag{38}$$

Since

$$y = r + u-c-1.5\dot{c} \tag{39}$$

(38) becomes

$$\ddot{c} + 2\dot{c} + 2c = 2r - \phi(y)r + \left\{2-\phi(y)\right\} u + \phi(y) \left\{c + 1.5\dot{c}\right\} \tag{40}$$

The equation for e then is

$$\ddot{e} + 2\dot{e} + 2e = \phi(y)r + \left\{\phi(y) - 2\right\} u - \phi(y) \left\{c + 1.5\dot{c}\right\} \tag{41}$$

The Liapunov function used in the last example is suitable here, the P matrix of (34) being applicable.

The control signal u required to insure a negative definite $\dot{V}(\underline{e})$ is found to be

$$u = \left\{\left|9(r-c)\right| + \left|15\dot{c}\right|\right\} \text{sat } b\ (2e_1 + 3e_2) \tag{42}$$

where again a b of 100 is used.

With this control signal applied, the overall system becomes asymptotically stable and can handle any magnitude of input signal. Analogue computer results indicated less than one percent error between plant and model outputs for step inputs as large as 10 volts. Fig. 10 shows a phase plane trajectory ($d'$ vs. $\dot{d}'$) for the system of Fig. 7 with a 4 volt input applied. The results for $d$ vs. $\dot{d}$ and $d'$ vs $\dot{d}'$ of Fig. 9, again for a 4 volt step input signal, are shown superimposed in Fig. 11. It is seen that a close agreement exists between plant and model outputs and output derivatives. The smooth curve is that of the model ($d$ vs $\dot{d}$). The noise in $\dot{d}'$ is attributable to the switching involved in generating u.

Since c and $c_d$ are in close agreement and the parts of the system between c and z and $c_d$ and x are linear, then signals x and z must also be nearly identical. This fact is used in design to determine over what range of y the restrictions on $f(y,t)$ must hold, and also to determine the required linear range in the feedback path which generates u. This is accomplished by determining what the maximum amplitude of x (also z) is for a given input, r. For step inputs of 2, 3, 4 and 5 volts, signals y and z were measured. Measured vs. theoretical values for y corresponding to each measured z are shown in Fig. 12. With this information it can be determined that for step inputs of 5 volts amplitude or less, the restrictions on $f(y,t)$ are required only for $|y| < 19$ (for example $f(y,t)$ might be represented by hard saturation for $|y| > 19$). Also in this case, the feedback path must be capable of handling $|u|_{max}$ without saturation. A conservative estimate for $|u|_{max}$ is 30 volts, and it is determined by the inequality

$$|u|_{max} < |y|_{max} + |r|_{max} + |c|_{max} \tag{43}$$

An important design consideration is the gain b in the linear range of the saturation function defined in (26c). Since sat $b\gamma$ is used rather than sign $\gamma$, there is a small region near the origin of the $e_1$, $e_2$ plane in which u may not be large enough to insure negative definiteness. Consequently, limit cycles may exist. However, by choosing b large enough, any such limit cycle can be reduced to a negligibly small amplitude.

## V. CONCLUSIONS

A nonlinear controller design procedure based on Liapunov's direct method and use of a model reference is extended in this paper to processes having not only time varying parameters, but a single time varying, nonlinear gain element as well. Thus, the procedure is applicable to a much wider class of control problems than heretofore. A condition which the nonlinearity must satisfy is derived. Inherent in the design procedure is the assurance that the resulting controller will have the desireable property of guaranteeing the asymptotic stability of the overall system.

Practical design considerations discussed in connection with a design example involving a conditionally stable process include the possibilities of limit cycles due to the function sat $b\gamma$, the method for determing capabilities of the feedback path, and the required range of $y$ for which $f(y,t)$ must satisfy the conditions imposed.

Analogue computer results presented bear out the validity of the theory and demonstrate the effectiveness of the controller in forcing the process output to track the model output.

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Appendix:

Definitions Pertinent to Liapunov's Second Method:

1. A function $F(\underline{x})$ is positive definite (negative definite) if $F = 0$ for $\underline{x} = 0$ and $F > 0$ $(F < 0)$ otherwise.

2. A function $G(\underline{x},t)$ is positive definite if there exists a positive definite function $F(\underline{x})$ such $G(\underline{x},t) \geq F(\underline{x})$ and $G(\underline{0}, t) = 0$.

Theorem I. The equilibrium state $\underline{x} = 0$ of the equation

$$\dot{\underline{x}} = A\underline{x} \tag{1}$$

where A is a constant matrix, is asymptotically stable if and only if, given any symetiric positive definite matrix Q, there exists a symmetric, positive definite matrix P such that

$$A^T P + PA = -Q \tag{2}$$

and $\underline{x}^T P \underline{x}$ is a Liapunov function for (1).

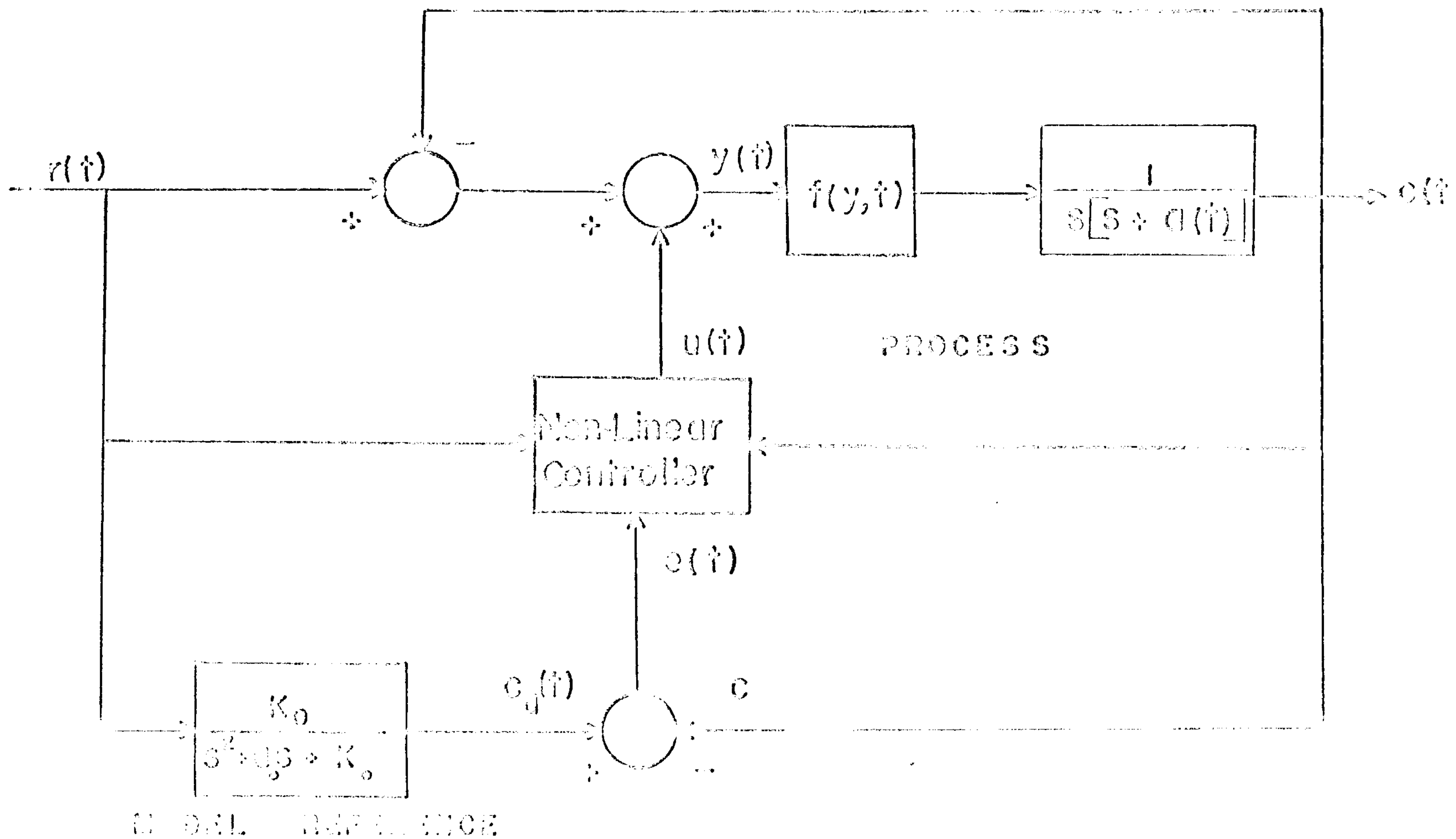



FIG. 1 SYSTEM BLOCK DIAGRAM

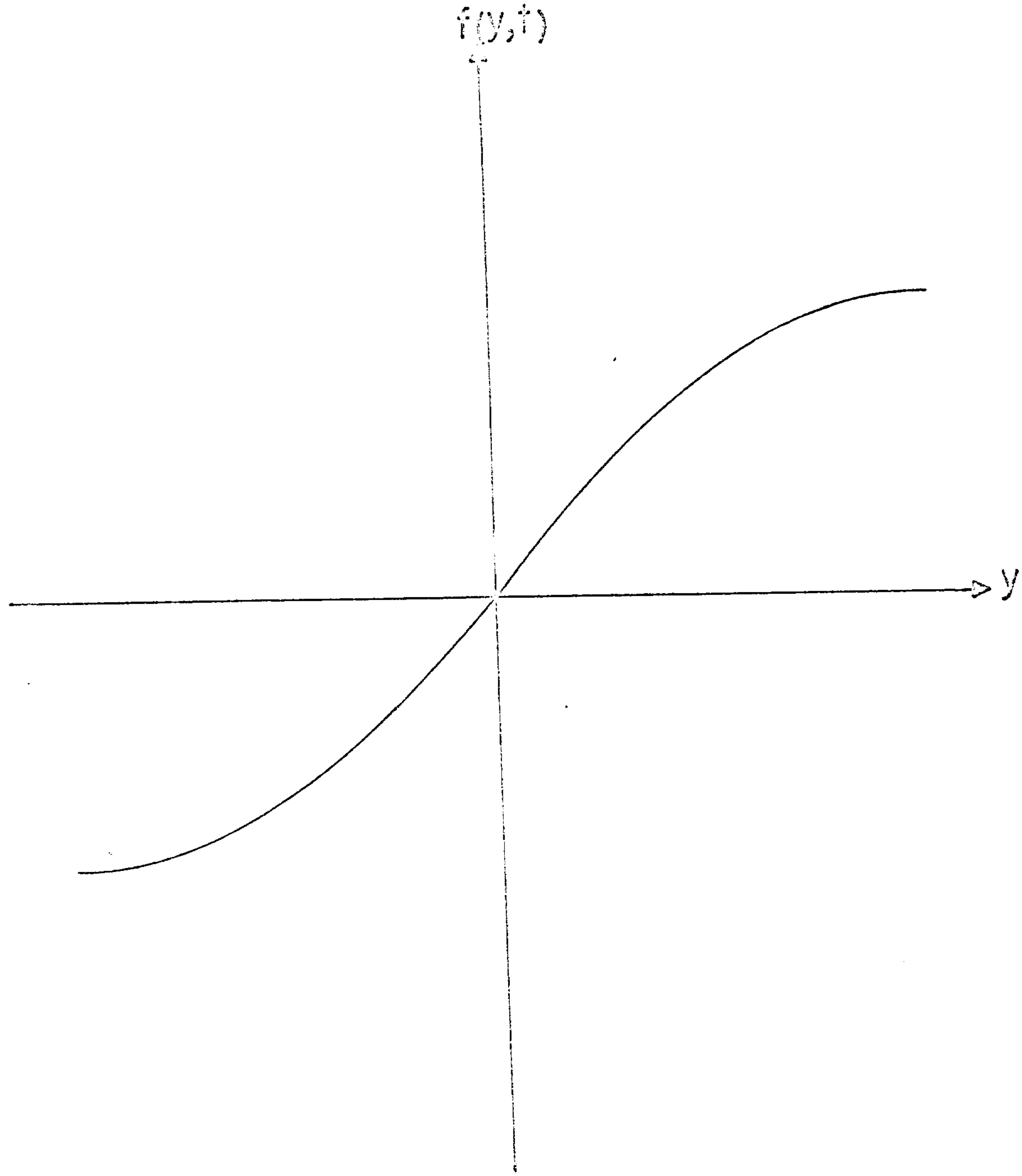
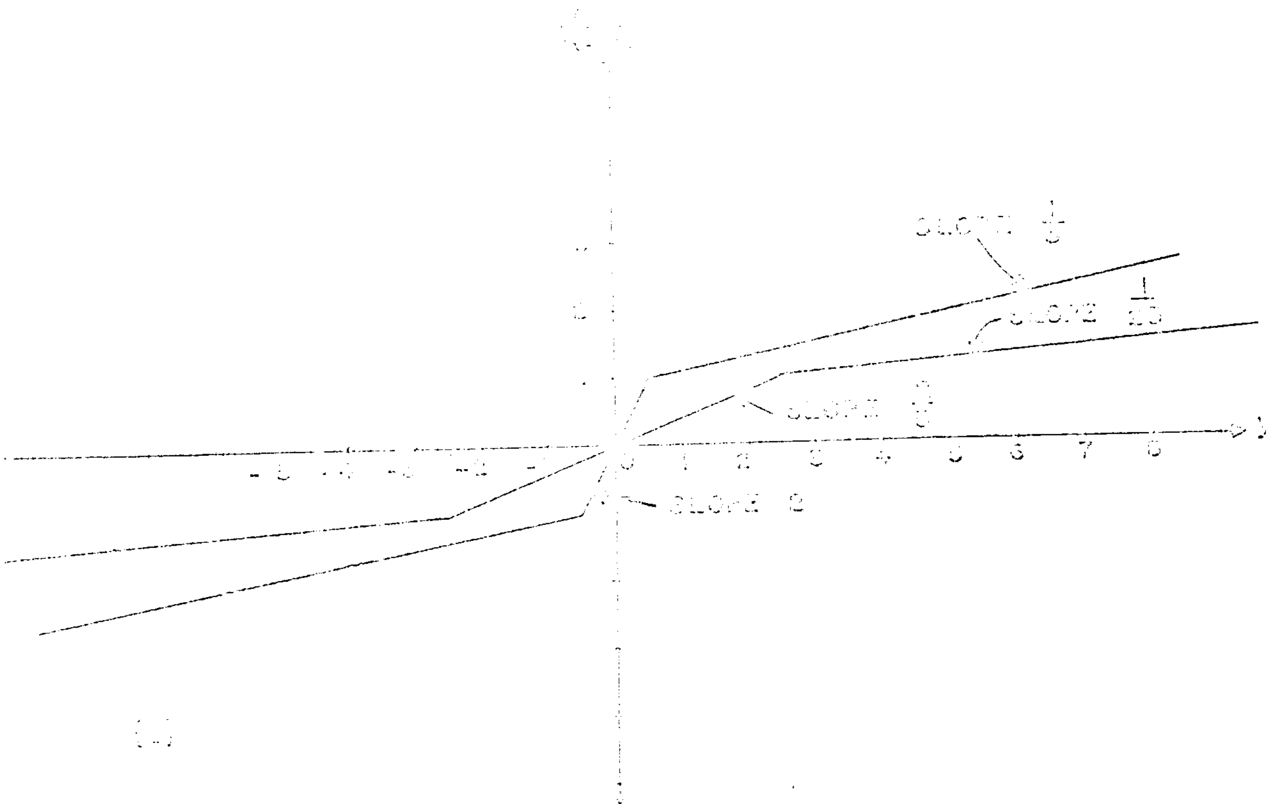



FIG. 2. GENERAL FORM OF NON-LINEAR FUNCTION.



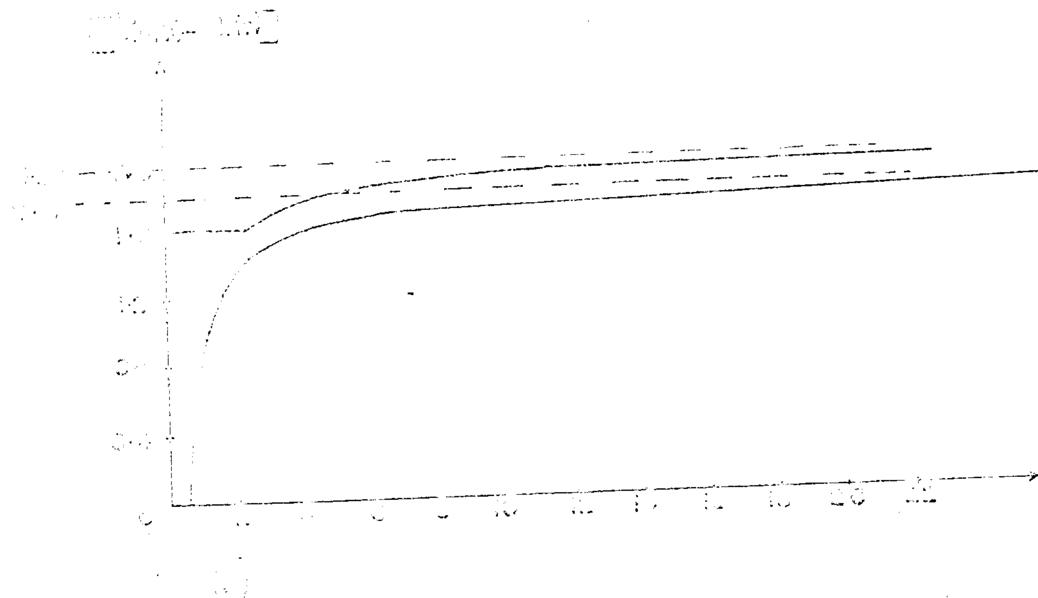



FIG. [illegible] FUNCTIONS FOR GENERAL EXAMPLE.
(a) Bounds on [illegible].
(b) Bounds on [illegible].

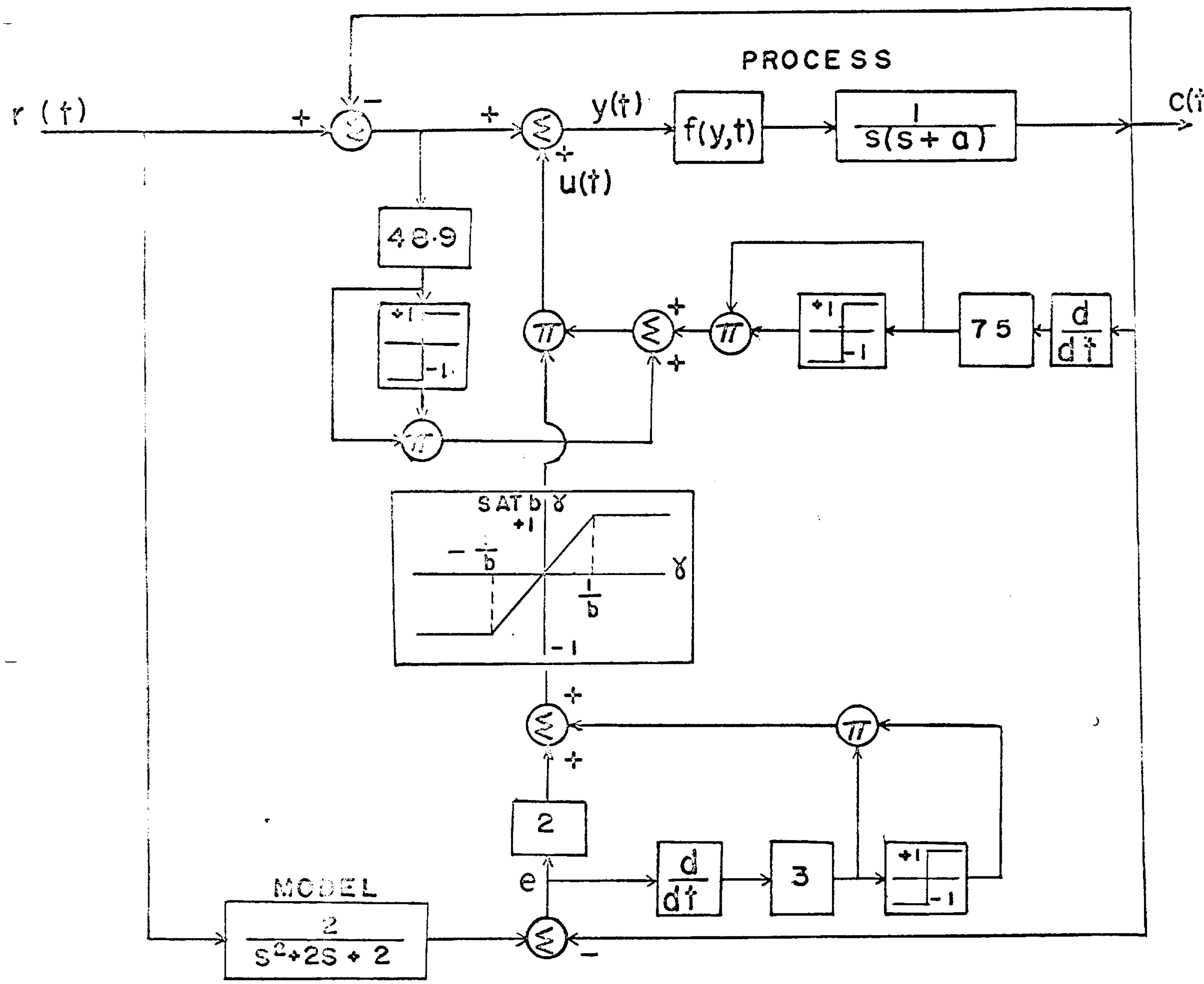
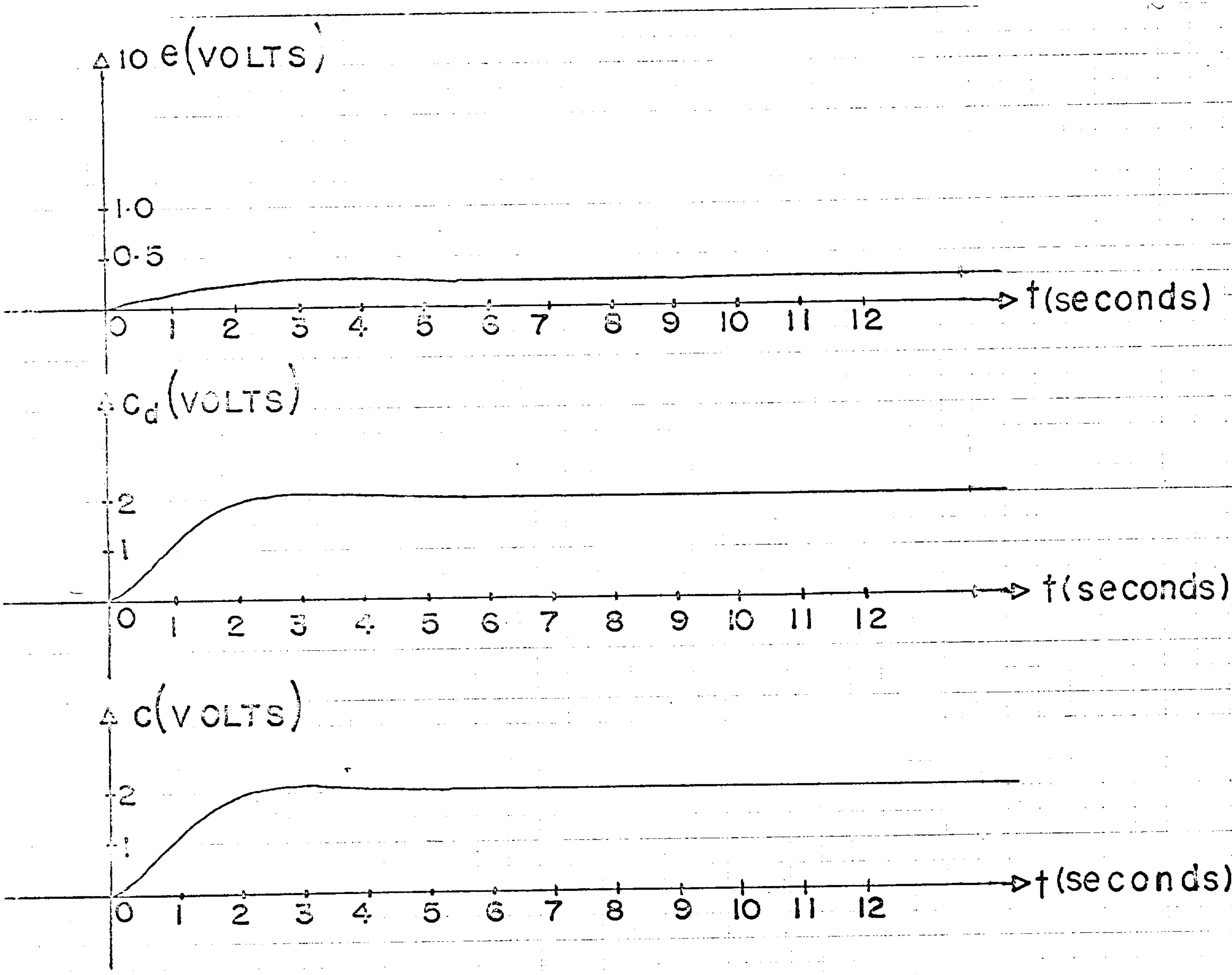



FIG. 4. SYSTEM FOR GENERAL EXAMPLE.




FIG· 5. STEP RESPONSE AND ERROR FOR SYSTEM OF FIG· 4.

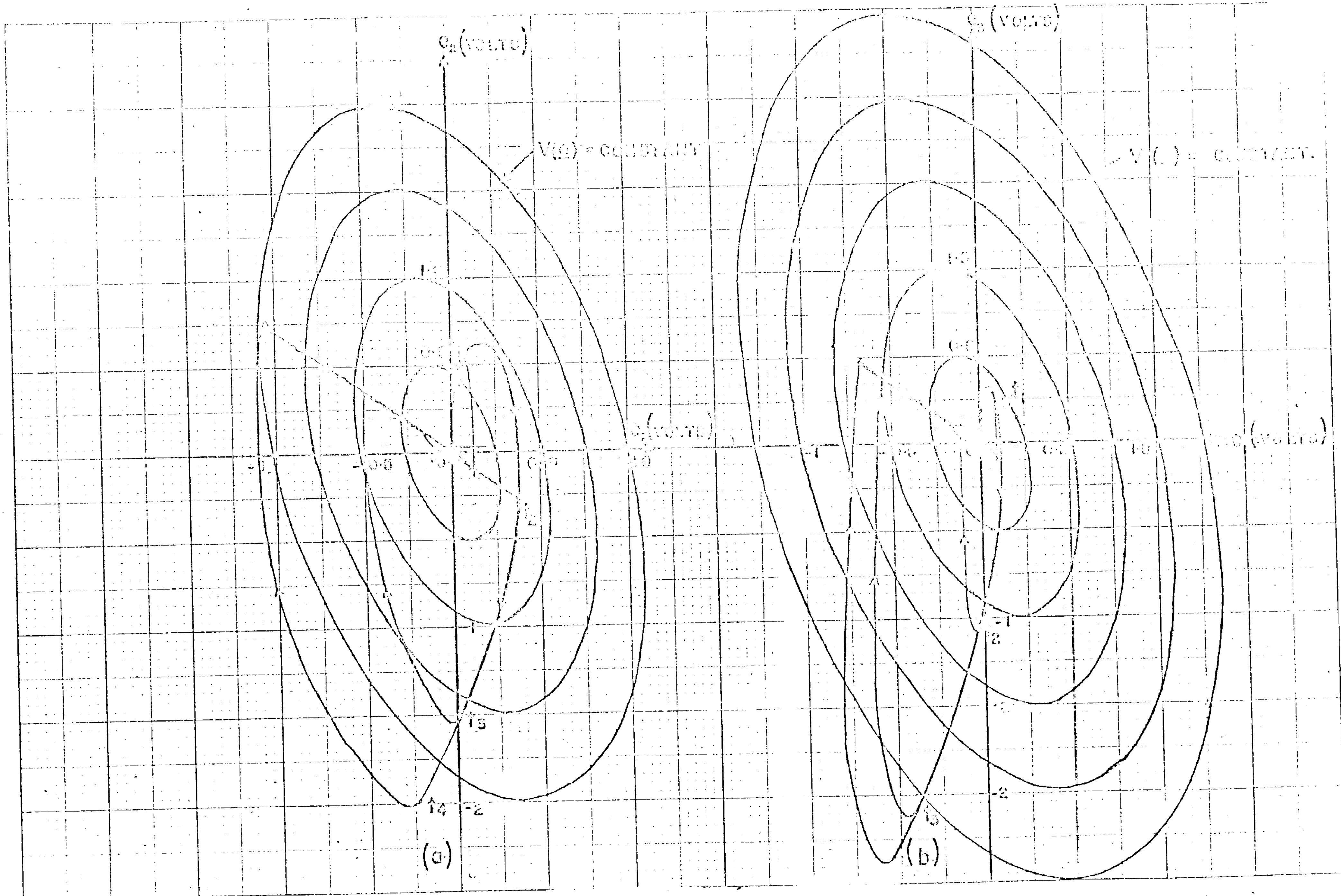



FIG. 6. TRAJECTORIES PROJECTED ON $e_1$, $e_2$ PLANE.

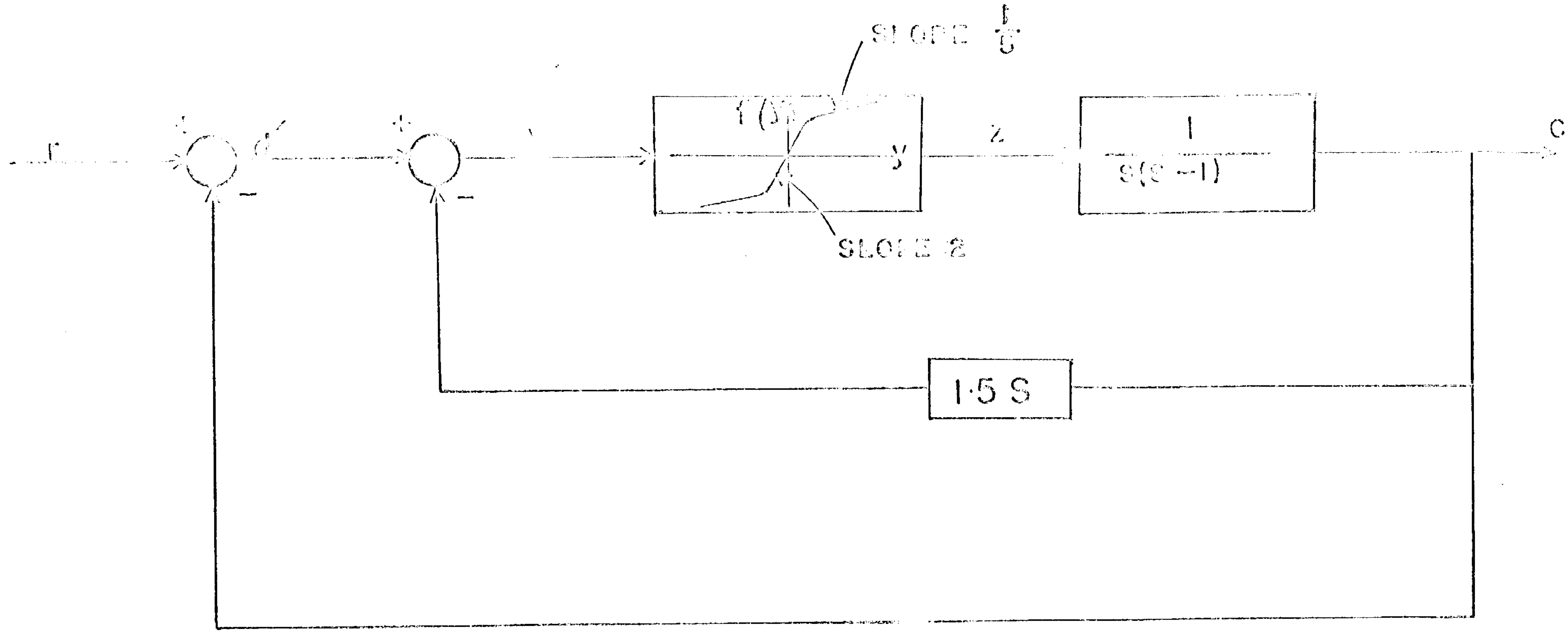



FIG. 7. CONDITIONALLY STABLE CONTROL SYSTEM.

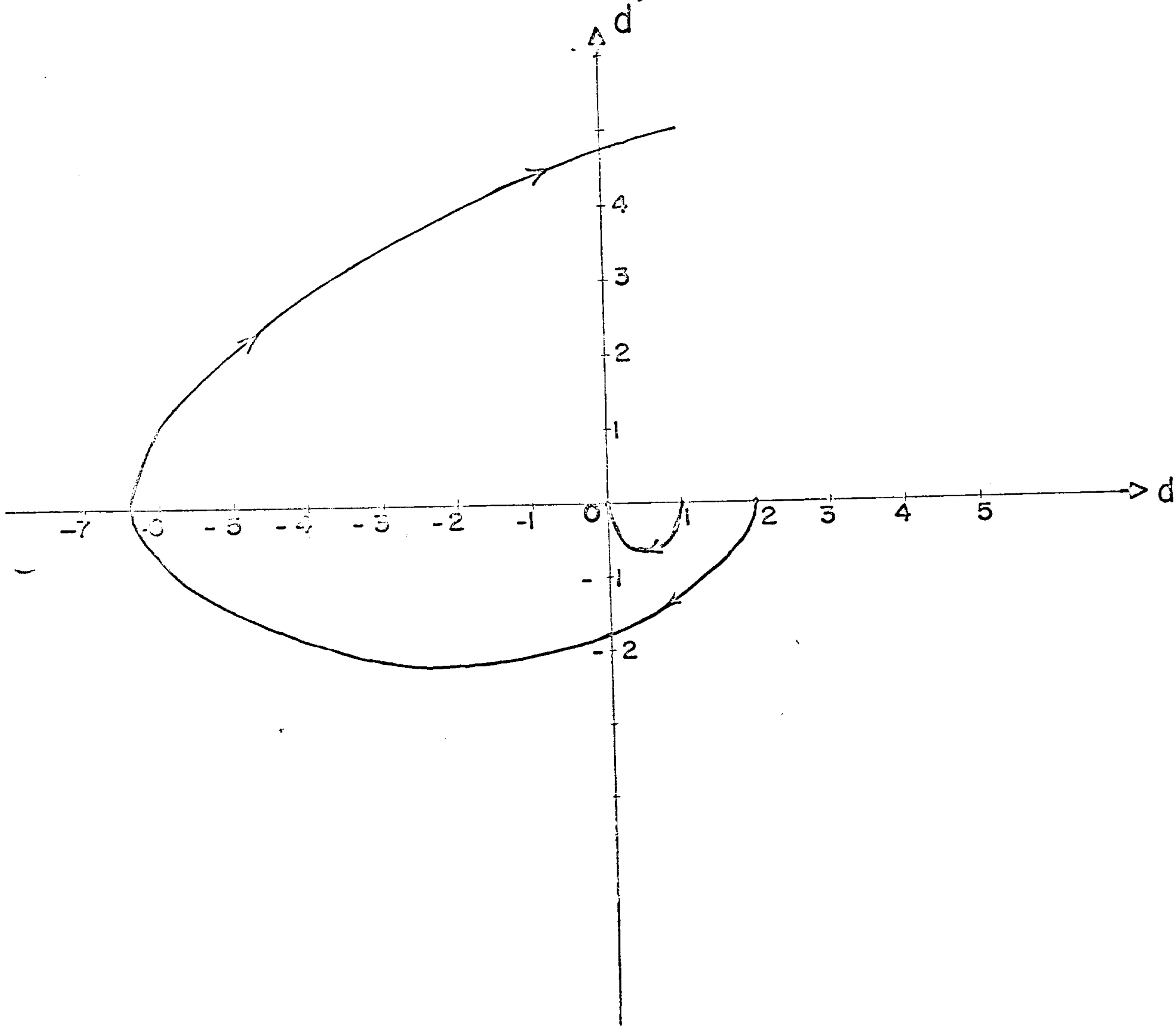
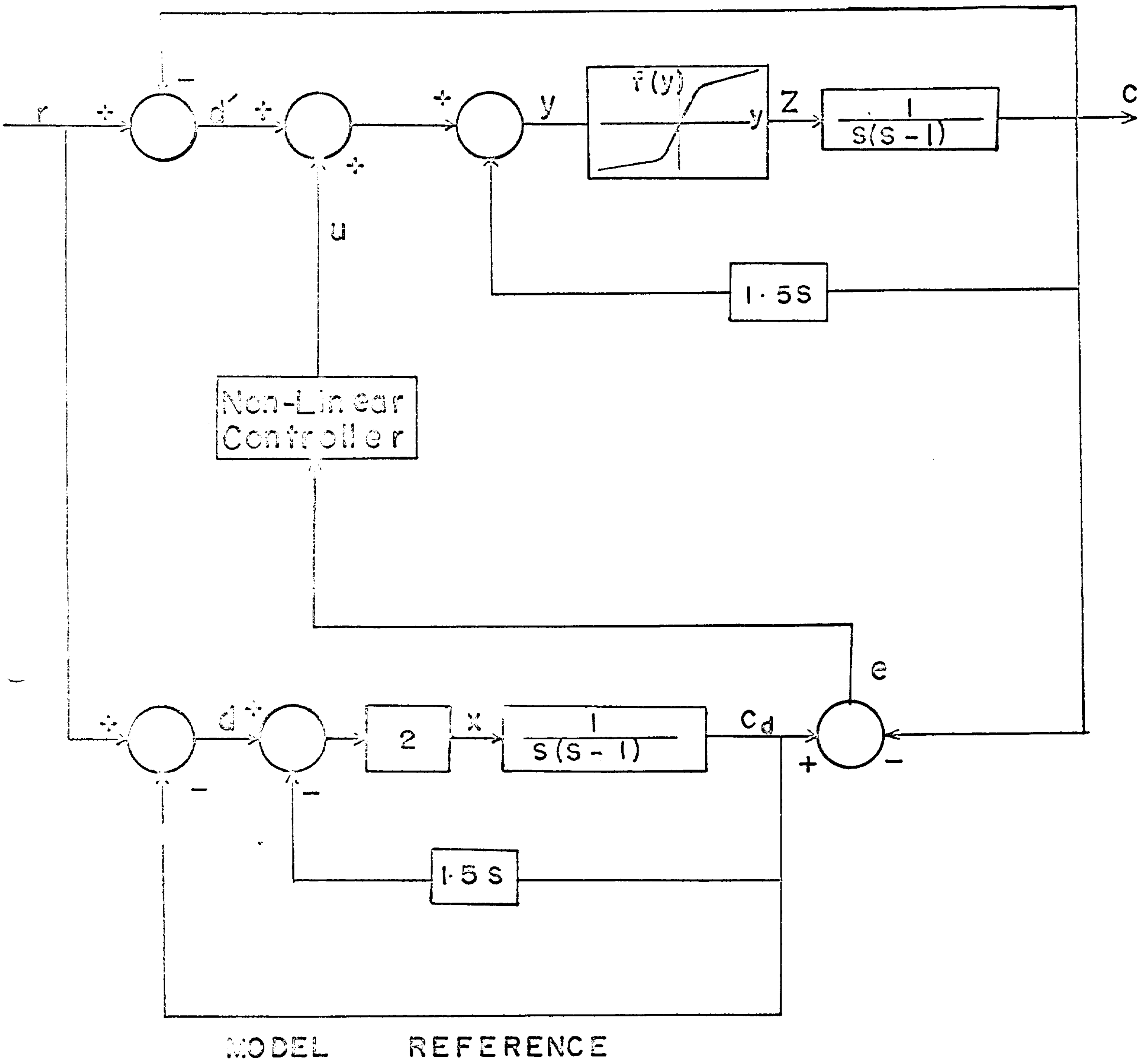
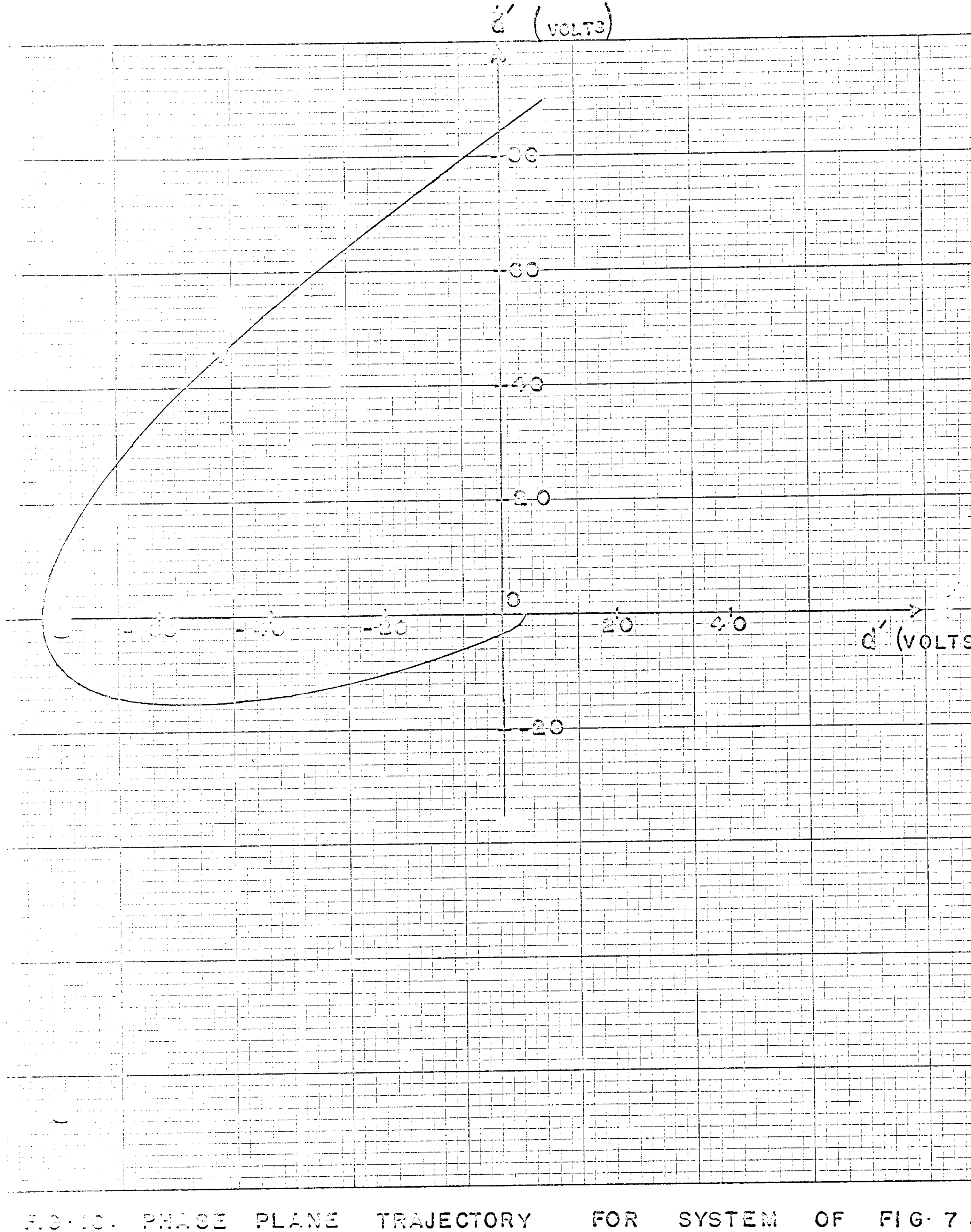



FIG. 8. PHASE PLANE TRAJECTORIES FOR SYSTEM OF FIG. 7.




FIG·9 NON LINEAR CONTROLLER APPLIED TO SYSTEM OF FIG· 7




F.3·10. PHASE PLANE TRAJECTORY FOR SYSTEM OF FIG· 7 .

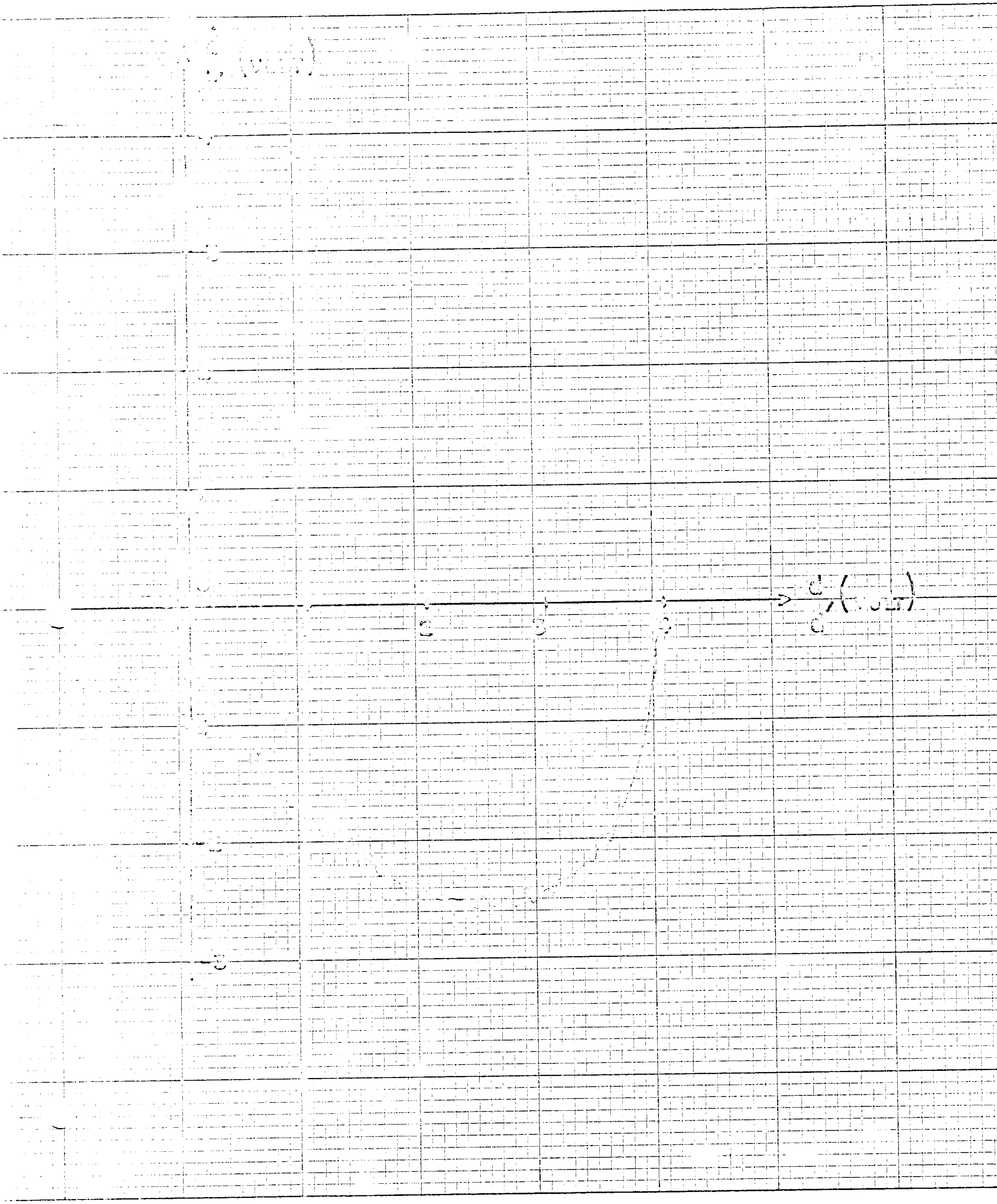



FIG. 11. PHASE PLANE TRAJECTORIES FOR SYSTEM OF FIG. 9.

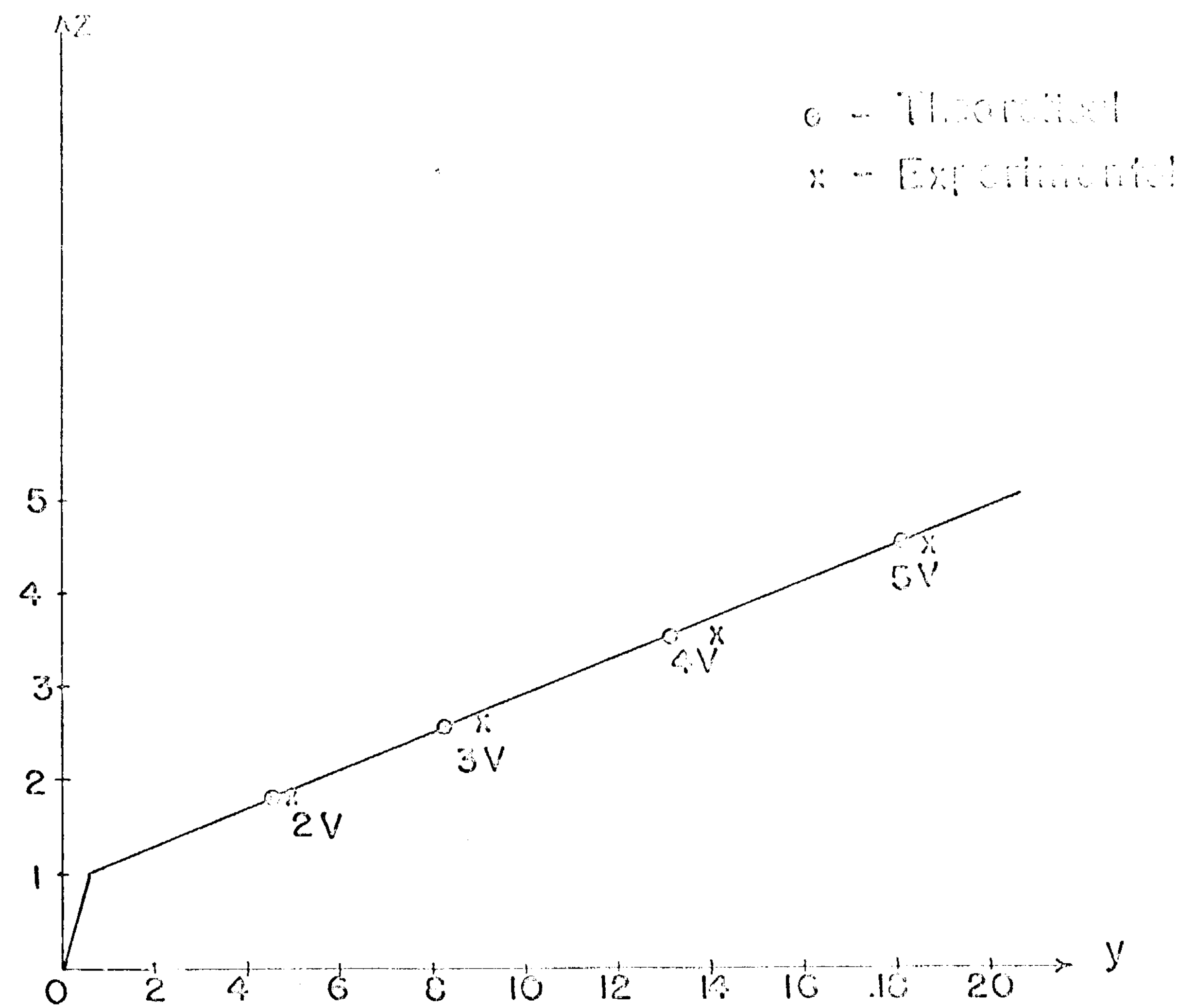



FIG · 12 — DESIGN DATA FOR SYSTEM OF FIG 9·